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On the available energy density for axisymmetric motions of a compressible stratified fluid

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An exact, local expression is obtained for the available energy density for axisymmetric motion of a compressible, stratified fluid. Under certain stated conditions the available energy density is positive definite; this fact can be used to demonstrate the stability of an appropriately defined reference state to finite-amplitude axisymmetric disturbances. The theory extends certain previous results on available energy and axisymmetric stability that are valid only for small-amplitude disturbances.

1. Introduction

This paper is concerned with a variant of the available potential energy (APE) concept of Lorenz (1955). The total potential energy (TPE) of a body of fluid is defined as the sum of the gravitational potential energy and the internal energy of the fluid. Lorenz considered a reference state of minimum TPE that could in principle be achieved, from an actual state, by a mass- and entropy-conserving redistribution of fluid 'particles'; however, this redistribution need not necessarily conserve dynamical properties such as angular momentum. The APE is defined as the difference between the TPE of the actual state and that of the state of minimum TPE.

The APE therefore sets an upper bound on the energy that is in some sense 'available', in a given fluid configuration, for conversion to kinetic energy in any subsequent motion that conserves mass and entropy. However, in the presence of further constraints on the subsequent motion, the maximum energy available for such conversion may well be less than the APE.

For small-amplitude departures of the flow from a statically stable reference state, Lorenz derived an expression for the APE in the form of an integral over the whole fluid, in which the integrand is manifestly positive definite. This integrand can be regarded as a locally defined 'available potential energy density'.

van Mieghem (1956) considered axisymmetric fluid motions that conserve mass, entropy, and angular momentum, and take the form of small departures from a steady axisymmetric reference state. He found an expression for the energy that is available for conversion into kinetic energy of the *meridional* motion in this case, in the form of an integral over the fluid. He showed that the integrand is positive definite when the reference state satisfies the conditions for symmetric stability. This integrand may be regarded as a local 'available energy density'; it includes a contribution associated with the kinetic energy of the zonal flow, in addition to a potential energy contribution.

Codoban & Shepherd (2003) similarly considered the effect of angular momentum conservation on the APE concept, but allowed for finite-amplitude disturbances from a rectilinear reference state for the case of a Boussinesq fluid on an 'f-plane'. They found a local form for the APE density that is positive definite when the reference state is symmetrically stable and presented a local conservation law for APE density that is valid even in the presence of eddy fluxes associated with zonally asymmetric processes.

The paper by Ilin (1991) derived conditions for the finite-amplitude symmetric stability of an axisymmetric baroclinic vortex, but did not consider the APE concept.

The present paper extends the results of Codoban & Shepherd (2003) to compressible flow with an axisymmetric reference state, using an approach similar to that of Andrews (1981) (hereafter referred to as A81). It also extends the results of van Mieghem (1956) to finite-amplitude disturbances, and provides stability conditions that have a clearer physical interpretation than those of Ilin (1991). The theory described here is relevant, for example, to axisymmetric flows in idealized models of terrestrial atmospheric and oceanic vortices, and of vortices in other planetary atmospheres. An extended version of the theory, briefly outlined in the Appendix, is relevant to zonally averaged atmospheric flows.

2. Basic equations

The equations for inviscid, adiabatic, compressible flow subject to a potential Φ are the momentum, continuity and entropy equations,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + \frac{1}{\rho}\nabla p + \nabla\Phi = 0, \tag{2.1}$$

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho\nabla\cdot\boldsymbol{u} = 0, \tag{2.2}$$

and

$$\frac{\mathrm{D}s}{\mathrm{D}t} = 0, \tag{2.3}$$

and the equation of state

$$\rho^{-1} = F(s, p). \tag{2.4}$$

Here $\boldsymbol{u}$ is the fluid velocity, ρ the density (assumed finite and non-zero everywhere), p the pressure and s the specific entropy; these are all functions of position $\boldsymbol{x}$ and time t. However, Φ is assumed to depend on position $\boldsymbol{x}$ only. $\mathrm{D}/\mathrm{D}t$ is the material derivative and F is a suitably smooth function. These equations are referred to an inertial frame; as noted in § 7 below, the extension to a rotating frame is straightforward.

From equations (2.1)–(2.4) the energy equation can be derived in the usual form

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left\{\tfrac{1}{2}\boldsymbol{u}^2 + \varepsilon + \Phi\right\} + \nabla\cdot(p\boldsymbol{u}) = 0, \tag{2.5}$$

where $\varepsilon(s, p)$ is the specific internal energy. The specific enthalpy $H(s, p) = \varepsilon + p/\rho$ and satisfies

$$H_s = T = G(s, p) \qquad \text{and} \qquad H_p = \rho^{-1} = F(s, p) \qquad \text{say,}$$

where subscripts s and p denote partial derivatives and G expresses the thermodynamic dependence of the temperature T on s and p; H will be assumed twice differentiable in s and p. Note that

$$\nabla H = T\nabla s + \rho^{-1}\nabla p = G\nabla s + F\nabla p. \tag{2.6}$$

We use cylindrical polar coordinates R, λ, z, where R is the perpendicular distance from the axis of symmetry, λ the azimuthal angle and z the axial distance. We define unit vectors $\boldsymbol{e}_R$, $\boldsymbol{e}_\lambda$, $\boldsymbol{e}_z$ in the coordinate directions. We put $\boldsymbol{u} = \boldsymbol{u}_\lambda + \boldsymbol{u}_{\mathrm{m}}$, where $\boldsymbol{u}_\lambda = \boldsymbol{e}_\lambda u$ is the zonal component of the flow and $\boldsymbol{u}_{\mathrm{m}}$ is the component of flow in the meridional plane.

The zonal angular momentum per unit mass is $m = uR$; by expressing (2.1) in cylindrical coordinates and taking the zonal component only, it can be shown that

$$\frac{\mathrm{D}m}{\mathrm{D}t} + \frac{1}{\rho}\frac{\partial p}{\partial \lambda} + \frac{\partial \Phi}{\partial \lambda} = 0. \tag{2.7}$$

This illustrates that m is materially conserved if the flow is zonally symmetric, when λ-derivatives are identically zero.

We define $\chi(R) \equiv 1/(2R^2)$ and note that $\nabla\chi = -R^{-3}\boldsymbol{e}_R$.

3. The reference state

We consider a steady, zonally symmetric reference state $\mathscr{R}_0$, in the form of a 'baroclinic circular vortex' with a purely zonal flow $\boldsymbol{u} = \boldsymbol{u}_0 = u_0(R,z)\,\boldsymbol{e}_\lambda$, $p = p_0(R,z)$, $\rho = \rho_0(R,z)$ and $\Phi = \Phi(R,z)$. Then (2.1) becomes

$$\boldsymbol{u}_0 \cdot \nabla\boldsymbol{u}_0 + \frac{1}{\rho_0}\nabla p_0 + \nabla\Phi = 0.$$

The first term equals the centripetal acceleration, and can be written as

$$\boldsymbol{u}_0 \cdot \nabla\boldsymbol{u}_0 = -\frac{u_0^2}{R}\boldsymbol{e}_R = -\frac{m_0^2}{R^3}\boldsymbol{e}_R = m_0^2\nabla\chi.$$

We put $\mu = m^2$, so that

$$\mu_0\nabla\chi + \rho_0^{-1}\nabla p_0 + \nabla\Phi = 0. \tag{3.1}$$

Applying (2.6) to $\mathscr{R}_0$ so as to eliminate $\rho_0^{-1}\nabla p_0$ we obtain

$$\mu_0\nabla\chi + \nabla H_0 - G_0\nabla s_0 + \nabla\Phi = 0$$

where $H_0 = H(s_0, p_0)$, and hence

$$\nabla(\mu_0\chi + H_0 + \Phi) = \chi\nabla\mu_0 + G_0\nabla s_0. \tag{3.2}$$

Then defining

$$C_0 \equiv -\left[\mu_0\chi + H(s_0, p_0) + \Phi\right] \tag{3.3}$$

(cf. Ilin 1991) we get

$$\nabla C_0 = -\chi\nabla\mu_0 - G_0\nabla s_0 \tag{3.4}$$

so C_0 is a function of μ_0 and s_0 only, say

$$C_0 = \tilde{C}(\mu_0, s_0). \tag{3.5}$$

From (3.4) we obtain

$$\chi = -\tilde{C}_\mu(\mu_0, s_0) \qquad \text{and} \qquad G_0 = T(s_0, p_0) = -\tilde{C}_s(\mu_0, s_0), \tag{3.6a, b}$$

where subscripts μ and s denote partial derivatives.

The vorticity of the reference flow is

$$\boldsymbol{\omega}_0 = \nabla \times \boldsymbol{u}_0 = R^{-1}(-m_{0z}\boldsymbol{e}_R + m_{0R}\boldsymbol{e}_z) = -R^{-1}\boldsymbol{e}_\lambda \times \nabla m_0$$

and the potential vorticity of the reference flow is

$$P_0 = \frac{\boldsymbol{\omega}_0 \cdot \nabla s_0}{\rho_0} = \frac{1}{\rho_0 R}\frac{\partial(m_0, s_0)}{\partial(R,z)} = \frac{1}{2m_0\rho_0 R}\frac{\partial(\mu_0, s_0)}{\partial(R,z)}. \tag{3.7}$$

Since $\chi = \chi(R)$, the scalar product of $\boldsymbol{e}_z$ with equation (3.1) implies that

$$\rho_0^{-1}\left(\frac{\partial p_0}{\partial z}\right)_R + \left(\frac{\partial \Phi}{\partial z}\right)_R = 0; \tag{3.8}$$

this expresses hydrostatic balance for the reference state.

Also, noting that $-\tilde{C}_{\mu s}(\mu_0, s_0) = \chi_s = G_{0\mu}$ from (3.6), we obtain

$$\left(\frac{\partial \chi}{\partial s_0}\right)_{\mu_0} = \left(\frac{\partial T_0}{\partial \mu_0}\right)_{s_0}. \tag{3.9}$$

Taking the scalar product of $\boldsymbol{e}_\lambda$ with the curl of (3.2) we obtain

$$\left(\frac{\partial \mu_0}{\partial z}\right)_R = R^3 \frac{\partial(T_0, s_0)}{\partial(R, z)} = 2m_0\rho_0 R^4 P_0 \left(\frac{\partial T_0}{\partial \mu_0}\right)_{s_0}, \tag{3.10}$$

after some manipulation of Jacobians and use of (3.7); we can regard (3.10) as the 'thermal windshear' equation for the reference state.

Since $\partial p_0/\partial t = 0$ the continuity equation implies

$$\nabla \cdot (p_0 \boldsymbol{u}) = \rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{p_0}{\rho}\right)$$

and so the energy equation (2.5) can be rewritten in the alternative form

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left\{\tfrac{1}{2}\boldsymbol{u}^2 + H - \frac{p - p_0}{\rho} + \Phi\right\} + \nabla \cdot [(p - p_0)\,\boldsymbol{u}] = 0: \tag{3.11}$$

see also A81 (equations (2.16)–(2.18)), and Bannon (2004, equation (2.7)).

4. Available energy for zonally symmetric flow

Consider now a zonally symmetric but time-varying flow. If friction and non-adiabatic effects are again neglected, the specific angular momentum and specific entropy are materially conserved:

$$\frac{\mathrm{D}\mu}{\mathrm{D}t} = 0, \qquad \frac{\mathrm{D}s}{\mathrm{D}t} = 0,$$

and hence any differentiable function of μ and s is also materially conserved. We shall consider the function $\tilde{C}(\mu, s)$ defined by equation (3.5); this is well-defined if the ranges of values of μ and s in the time-varying flow are contained within those in the reference state $\mathscr{R}_0$. If this is not the case, we assume that a suitable analytic continuation of $\tilde{C}$ can be found. The volume integral $\int \rho\tilde{C}\,\mathrm{d}V$, taken over the fluid domain, is constant in time and is a *Casimir invariant*: see e.g. Shepherd (1993).

From (3.3) and (3.5) we have

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} = -\frac{\mathrm{D}}{\mathrm{D}t}\left\{\tilde{C}(\mu_0, s_0) + \mu_0\chi + H(s_0, p_0)\right\}.$$

Adding $\mathrm{D}\tilde{C}(\mu, s)/\mathrm{D}t \equiv 0$ to the right of this equation and using (3.6*a*) we obtain

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\left\{\tilde{C}(\mu, s) - \tilde{C}(\mu_0, s_0) + \mu_0\tilde{C}_\mu(\mu_0, s_0) - H(s_0, p_0)\right\}. \tag{4.1}$$

The kinetic energy per unit mass is

$$\tfrac{1}{2}\boldsymbol{u}^2 = \tfrac{1}{2}\left(u^2 + \boldsymbol{u}_{\mathrm{m}}^2\right) = \tfrac{1}{2}\left(\frac{m^2}{R^2} + \boldsymbol{u}_{\mathrm{m}}^2\right) = \mu\chi + \tfrac{1}{2}\boldsymbol{u}_{\mathrm{m}}^2 = -\mu\tilde{C}_\mu(\mu_0, s_0) + \tfrac{1}{2}\boldsymbol{u}_{\mathrm{m}}^2, \tag{4.2}$$

where (3.6*b*) has been used in the last expression.

Combining equations (3.11), (4.1) and (4.2) we therefore obtain the following version of the energy equation:

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\tfrac{1}{2}\boldsymbol{u}_{\mathrm{m}}^2 + A\right) + \nabla\cdot[(p-p_0)\,\boldsymbol{u}] = 0, \tag{4.3}$$

where the *available energy per unit mass* A is defined by

$$A \equiv \tilde{C}(\mu, s) - \tilde{C}(\mu_0, s_0) - (\mu-\mu_0)\tilde{C}_\mu(\mu_0, s_0) + H(s, p) - H(s_0, p_0) - \frac{p-p_0}{\rho}. \tag{4.4}$$

Clearly A is identically zero at each point in the reference state; we shall now show that, under certain conditions, it is positive definite for all states. We first put A into a more convenient form $A = A_1 + A_2$ by subtracting and adding a term $(s-s_0)\tilde{C}_s(\mu_0, s_0)$, as follows:

$$A_1 \equiv \tilde{C}(\mu, s) - \tilde{C}(\mu_0, s_0) - (\mu-\mu_0)\tilde{C}_\mu(\mu_0, s_0) - (s-s_0)\tilde{C}_s(\mu_0, s_0),$$

$$A_2 \equiv H(s, p) - H(s_0, p_0) - \frac{p-p_0}{\rho} + (s-s_0)\tilde{C}_s(\mu_0, s_0).$$

The term A_1 is in the form noted by Shepherd (1993, equation (10.8)); by Taylor's theorem (including the explicit remainder) it is positive definite, even for finite values of $\mu-\mu_0$ and $s-s_0$, provided that

$$\tilde{C}_{\mu\mu} > 0, \qquad \tilde{C}_{ss} > 0, \qquad \tilde{C}_{\mu\mu}\tilde{C}_{ss} - \tilde{C}_{\mu s}^2 > 0 \tag{4.5}$$

where subscripts μ and s denote partial derivatives. Note that Taylor's theorem in this form requires conditions (4.5) to hold at some values μ_1, s_1 say, intermediate between μ_0, μ and s_0, s, respectively; this in turn requires that the domain in the (μ, s)-plane where $C(\mu, s)$ is defined must be convex. We shall assume that these conditions hold for all values of these variables encountered by the flow. Since $\tilde{C}_{\mu\mu}$, $\tilde{C}_{ss}$ and $\tilde{C}_{\mu\mu}\tilde{C}_{ss} - \tilde{C}_{\mu s}^2$ are functions of μ and s alone, their values – and hence their signs – are constant in time at each point in (μ, s)-space. If the conditions (4.5) hold at any instant throughout the flow, they therefore hold for all times.

We next split the term A_2 into two further expressions, $A_2 = A_{21} + A_{22}$:

$$A_{21} \equiv H(s, p) - H(s, p_0) - \frac{p-p_0}{\rho}, \qquad A_{22} \equiv H(s, p_0) - H(s_0, p_0) - (s-s_0)G(s_0, p_0),$$

where (3.6*b*) has been used again in the final term on the right. The term A_{21} is identical to the expression for Π_1 of A81, and hence can be written in the form (A81, equation (3.4))

$$A_{21} = -\int_{p_0}^{p} \left(p' - p_0\right) H_{pp}(s, p')\,\mathrm{d}p'.$$

This is positive definite provided that $H_{pp} = F_p = -(\rho c)^{-2} < 0$, where

$$c^2(s, p) \equiv \left(\frac{\partial p(\rho, s)}{\partial \rho}\right)_{\rho^{-1}=F(s,p)},$$

or equivalently

$$0 < c^2 < \infty, \tag{4.6}$$

where $c(s, p)$ is the speed of sound. The term A_{22} is not identical to Π_2 of A81, but can be written

$$A_{22} = H(s, p_0) - H(s_0, p_0) - (s - s_0)H_s(s_0, p_0)$$
$$= \int_{s_0}^{s} \left[H_s(s', p_0) - H_s(s_0, p_0)\right] \mathrm{d}s' = \int_{s_0}^{s} \left[G(s', p_0) - G(s_0, p_0)\right] \mathrm{d}s',$$

and this is positive definite provided that $G_s = H_{ss} = (\partial T/\partial s)_p \equiv T/c_\mathrm{p} > 0$ where c_p is the specific heat capacity at constant pressure. This condition therefore holds for gases for which

$$c_\mathrm{p} > 0 \tag{4.7}$$

and in particular for an ideal gas: see § 5.1 below.

In summary, the available energy per unit mass A is positive definite at each point where the conditions (4.5), (4.6) and (4.7) hold.

It should be noted that Ilin (1991), in a study of the finite-amplitude symmetric stability of a baroclinic vortex, gave expressions equivalent to our A_1 and A_2 but did not split A_2 in the same manner as done here. He provided conditions for the positive definiteness of A that are slightly different from ours and that do not lend themselves to such simple physical interpretations as (4.6) and (4.7) above.

5. Special cases

5.1. *Results for an ideal gas*

In the case of an ideal gas, we have (see A81)

$$\rho^{-1} = F(s, p) = (\kappa c_\mathrm{p})\mathrm{e}^{s/c_\mathrm{p}}\, p^{-(1-\kappa)}, \quad T = G(s, p) = \mathrm{e}^{s/c_\mathrm{p}}\, p^{\kappa}, \quad H = c_\mathrm{p}\mathrm{e}^{s/c_\mathrm{p}}\, p^{\kappa}, \tag{5.1a–c}$$

where c_p is a positive constant, and κ is the specific gas constant divided by c_p and lies between 0 and 1. For convenience the pressure has been normalized by a standard value.

As in A81 we have

$$A_{21} = c_\mathrm{p}\mathrm{e}^{s/c_\mathrm{p}}\, p_0^{\kappa}\, f(p/p_0) \qquad \text{where} \qquad f(\xi) \equiv (1-\kappa)\xi^{\kappa} + \kappa\xi^{-(1-\kappa)} - 1.$$

For $\xi > 0$ it can be shown that

$$f(1) = 0; \qquad f(\xi) > 0 \quad \text{for} \quad \xi \neq 1$$

(using $0 < \kappa < 1$). This verifies that $A_{21} > 0$ for $p \neq p_0$.

It can also be shown that

$$A_{22} = c_\mathrm{p}\mathrm{e}^{s_0/c_\mathrm{p}}\, p_0^{\kappa}\, h((s - s_0)/c_\mathrm{p}) \qquad \text{where} \qquad h(\eta) \equiv \mathrm{e}^{\eta} - 1 - \eta. \tag{5.2}$$

It can be shown that $h(\eta) > 0$ for all non-zero η, verifying that $A_{22} > 0$ for $s \neq s_0$.

5.2. *Small-amplitude disturbances: comparison with previous results*

In this section we show that, for small-amplitude disturbances of an ideal gas, our expression for A reduces to that derived by Fjørtoft (1946, 1950), and discussed by Eliassen & Kleinschmidt (1957, pp. 66–70) and Charney (1973, pp. 142–167), in the context of the conditions for symmetric stability of an ideal gas. The connection between the stability criterion and the sign-definiteness of small-amplitude expressions for 'available energy' was apparently first made by van Mieghem (1956).

These authors used a Lagrangian approach, considering infinitesimal meridional displacements $\delta\boldsymbol{r}$ of fluid particles from an undisturbed state, $\mathscr{S}_0$ say, to a disturbed state $\mathscr{S}$. For convenience we follow Charney (1973, p. 149), who showed that symmetric stability of $\mathscr{S}_0$ is assured if the quadratic form (in our notation)

$$I = I_1 + I_2$$

is positive definite for displacements that conserve mass, specific entropy and specific angular momentum, where

$$I_1 \equiv -\tfrac{1}{2}\delta \boldsymbol{r} \cdot \left(\nabla\mu_0 \nabla\chi + \frac{1}{\rho_0 c_\mathrm{p}} \nabla s_0 \nabla p_0 \right) \cdot \delta \boldsymbol{r} \tag{5.3}$$

and

$$I_2 \equiv \frac{1}{2\rho_0^2 c_0^2} \left(\delta p - \delta \boldsymbol{r} \cdot \nabla p_0\right)^2 . \tag{5.4}$$

Here δp is the (Lagrangian) change in pressure experienced by a particle during the small displacement $\delta \boldsymbol{r}(\boldsymbol{x})$ from its initial position $\boldsymbol{x}$ in $\mathscr{S}_0$. Hence

$$p(\boldsymbol{x} + \delta \boldsymbol{r}) = p_0(\boldsymbol{x}) + \delta p(\boldsymbol{x}). \tag{5.5}$$

Clearly the corresponding Lagrangian changes $\delta\mu$ and δs of the materially conserved quantities μ and s are zero.

These Lagrangian changes should be contrasted with the 'Eulerian' differences *at fixed points* between quantities in the disturbed state $\mathscr{S}$ and the undisturbed state $\mathscr{S}_0$. Consider for example the Eulerian difference in pressure

$$\Delta p(\boldsymbol{x}) \equiv p(\boldsymbol{x}) - p_0(\boldsymbol{x}).$$

Using (5.5),

$$\Delta p(\boldsymbol{x}) = p(\boldsymbol{x}) - p(\boldsymbol{x} + \delta \boldsymbol{r}) + \delta p(\boldsymbol{x}) \approx \delta p(\boldsymbol{x}) - \delta \boldsymbol{r} \cdot \nabla p(\boldsymbol{x})$$

to leading order in the displacement amplitude. But since $\mathscr{S}$ is only a small departure from $\mathscr{S}_0$ we can replace p by p_0 in the last term, to the same approximation, to obtain

$$\Delta p(\boldsymbol{x}) = \delta p(\boldsymbol{x}) - \delta \boldsymbol{r} \cdot \nabla p_0(\boldsymbol{x}) \tag{5.6}$$

to leading order. Since $\delta\mu = \delta s = 0$ we also have

$$\Delta\mu = -\delta \boldsymbol{r} \cdot \nabla\mu_0 \quad \text{and} \quad \Delta s(\boldsymbol{x}) = -\delta \boldsymbol{r} \cdot \nabla s_0(\boldsymbol{x}). \tag{5.7}$$

Using these results we find

$$2I_1 = \Delta\mu \left(\delta \boldsymbol{r} \cdot \nabla\chi\right) + \frac{\Delta s}{\rho_0 c_\mathrm{p}} \left(\delta \boldsymbol{r} \cdot \nabla p_0\right).$$

We now assume that

$$J_0 \equiv \frac{\partial(\mu_0, s_0)}{\partial(R, z)} \neq 0, \tag{5.8}$$

so that $\chi(R)$ and $p_0(R, z)$ can alternatively be regarded as functions of μ_0 and s_0. (Note that (5.8) implies $m_0 P_0 \neq 0$, by (3.7).) We can therefore write

$$\nabla\chi = \left(\frac{\partial\chi}{\partial\mu_0}\right)_{s_0} \nabla\mu_0 + \left(\frac{\partial\chi}{\partial s_0}\right)_{\mu_0} \nabla s_0$$

and a similar expression for ∇p_0 and use equations (5.7) to obtain

$$2I_1 = -\Delta\mu \left[\left(\frac{\partial\chi}{\partial\mu_0}\right)_{s_0} \Delta\mu + \left(\frac{\partial\chi}{\partial s_0}\right)_{\mu_0} \Delta s \right] - \frac{\Delta s}{\rho_0 c_\mathrm{p}} \left[\left(\frac{\partial p_0}{\partial\mu_0}\right)_{s_0} \Delta\mu + \left(\frac{\partial p_0}{\partial s_0}\right)_{\mu_0} \Delta s \right].$$

On tidying up, this gives

$$I_1 = -\tfrac{1}{2}\left(\frac{\partial \chi}{\partial \mu_0}\right)_{s_0}(\Delta\mu)^2 - \tfrac{1}{2}\left[\frac{1}{\rho_0 c_{\mathrm{p}}}\left(\frac{\partial p_0}{\partial \mu_0}\right)_{s_0} + \left(\frac{\partial \chi}{\partial s_0}\right)_{\mu_0}\right]\Delta\mu\,\Delta s - \frac{1}{2\rho_0 c_{\mathrm{p}}}\left(\frac{\partial p_0}{\partial s_0}\right)_{\mu_0}(\Delta s)^2 .$$

Moreover, by equations (3.9), (5.1) and the ideal gas law it can be verified that

$$\frac{1}{\rho_0 c_{\mathrm{p}}}\left(\frac{\partial p_0}{\partial \mu_0}\right)_{s_0} = \left(\frac{\partial \chi}{\partial s_0}\right)_{\mu_0},$$

so that

$$I_1 = -\tfrac{1}{2}\left(\frac{\partial \chi}{\partial \mu_0}\right)_{s_0}(\Delta\mu)^2 - \left(\frac{\partial \chi}{\partial s_0}\right)_{\mu_0}\Delta\mu\,\Delta s - \frac{1}{2\rho_0 c_{\mathrm{p}}}\left(\frac{\partial p_0}{\partial s_0}\right)_{\mu_0}(\Delta s)^2 . \tag{5.9}$$

Using the standard conditions for positive definiteness of a quadratic form and some manipulation of Jacobians, it can be shown that $I_1 \geqslant 0$ for all $\Delta\mu$, Δs if

$$\frac{\partial(\chi, p_0)}{\partial(\mu_0, s_0)} > 0 \quad \text{and} \quad \left(\frac{\partial \mu_0}{\partial R}\right)_{s_0} > 0. \tag{5.10}$$

Using the hydrostatic equation (3.8) and (5.8), the first of inequalities (5.10) leads to

$$\rho_0 J_0 \left(\frac{\partial \Phi}{\partial z}\right)_R > 0. \tag{5.11}$$

Moreover it can be shown by manipulation of Jacobians that

$$J_0 = \left(\frac{\partial \mu_0}{\partial R}\right)_{s_0}\left(\frac{\partial s_0}{\partial z}\right)_R .$$

A set of sufficient conditions for I_1 to be positive definite is therefore

$$\left(\frac{\partial \Phi}{\partial z}\right)_R > 0, \qquad \left(\frac{\partial \mu_0}{\partial R}\right)_{s_0} > 0, \qquad \left(\frac{\partial s_0}{\partial z}\right)_R > 0. \tag{5.12}$$

These correspond, respectively, to the potential increasing in the z-direction (as in the case of gravity acting in the negative z-direction), the squared angular momentum increasing in the radial direction on isentropes (Rayleigh's stability criterion) and the entropy increasing in the z-direction (static stability): see e.g. Charney (1973). Using (5.11) these conditions also imply that $J_0 > 0$ and hence, by (3.7), that $m_0 P_0 > 0$, i.e. that the product of the angular momentum per unit mass and the potential vorticity of the reference flow is strictly positive. (Note, incidentally, that the first and third of conditions (5.12) can be combined in the form

$$\left(\frac{\partial s_0}{\partial \Phi}\right)_R > 0,$$

i.e. the specific entropy increases with increasing potential at constant R: this form is convenient for extensions to spherical geometry, for example.)

Furthermore using (5.6) we can rewrite I_2 simply as

$$I_2 = \frac{1}{2\rho_0^2 c_0^2}(\Delta p)^2 . \tag{5.13}$$

This is clearly positive definite provided $c_0^2 > 0$.

The sum $I_1 + I_2$ is therefore positive definite under the stated conditions, and this implies stability of the reference flow to small-amplitude axisymmetric disturbances. Further details are given by Charney (1973), for example.

Moreover $\rho_0(I_1 + I_2)$ can be identified with the integrand in van Mieghem's small-amplitude expression (12.5) for the 'available energy for conversion into kinetic energy of the two-dimensional motion'. Van Mieghem noted that this available energy is positive when the fluid equilibrium state is stable.

We now compare the expressions (5.3) and (5.4), or equivalently (5.9) and (5.13), with the results of § 4 by considering the case of small disturbances from the reference state $\mathscr{R}_0$, and evaluating our expressions for A_1, A_{21} and A_{22} to second order in the disturbance quantities $\Delta\mu \equiv \mu - \mu_0$, $\Delta s \equiv s - s_0$ and $\Delta p \equiv p - p_0$, all assumed small. We continue to assume that the fluid is an ideal gas.

For A_1 we obtain the usual quadratic terms in a Taylor expansion,

$$A_1 \approx \tfrac{1}{2}\tilde{C}_{\mu\mu}(\mu_0, s_0)(\Delta\mu)^2 + \tilde{C}_{\mu s}(\mu_0, s_0)\Delta\mu\,\Delta s + \tfrac{1}{2}\tilde{C}_{ss}(\mu_0, s_0)(\Delta s)^2. \tag{5.14}$$

From (3.6*a*) we obtain

$$\tilde{C}_{\mu\mu}(\mu_0, s_0) = -\left(\frac{\partial\chi}{\partial\mu_0}\right)_{s_0} \quad \text{and} \quad \tilde{C}_{\mu s}(\mu_0, s_0) = -\left(\frac{\partial\chi}{\partial s_0}\right)_{\mu_0}. \tag{5.15}$$

On the other hand, from (3.6*b*) we obtain

$$\tilde{C}_{ss}(\mu_0, s_0) = -\left(\frac{\partial T(s_0, p_0)}{\partial s_0}\right)_{\mu_0} = -\left(\frac{\partial T_0}{\partial s_0}\right)_{p_0} - \left(\frac{\partial T_0}{\partial p_0}\right)_{s_0}\left(\frac{\partial p_0}{\partial s_0}\right)_{\mu_0}$$

and using (5.1*a–c*) this gives, for an ideal gas,

$$\tilde{C}_{ss}(\mu_0, s_0) = -\frac{T_0}{c_p} - \frac{1}{\rho_0 c_p}\left(\frac{\partial p_0}{\partial s_0}\right)_{\mu_0}. \tag{5.16}$$

As in A81, the small-amplitude expression for A_{21} is

$$A_{21} \approx \frac{1}{2\rho_0^2 c_0^2}(\Delta p)^2, \tag{5.17}$$

where c_0 is the sound speed in the reference state. The corresponding expression for A_{22} follows from (5.2) and (5.1*b*), using the fact that $h(\eta) \approx \eta^2/2$ for small η:

$$A_{22} \approx \frac{T_0}{2c_p}(\Delta s)^2. \tag{5.18}$$

Combining equations (5.14)–(5.18) and comparing with equations (5.9) and (5.13) we see that

$$A_1 + A_{22} \approx I_1 \quad \text{and} \quad A_{21} \approx I_2,$$

thus verifying that the finite-amplitude expressions of § 4 reduce to those of Fjørtoft and others in the small-amplitude limit. It should be observed that terms in $(\Delta s)^2$ appear in both A_1 and A_2.

6. Stability to finite-amplitude axisymmetric disturbances

We integrate (4.3) over the fluid volume, V say, assuming that the boundary of V is rigid so that $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ there, where $\boldsymbol{n}$ is the unit normal. Using the continuity equation (2.2) this gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\left(\tfrac{1}{2}\boldsymbol{u}_{\mathrm{m}}^2 + A\right)\,\mathrm{d}V = 0.$$

Defining the volume-integrated meridional kinetic energy $\mathscr{M}$ and available energy $\mathscr{A}$, respectively, by

$$\mathscr{M}(t) \equiv \int_V \tfrac{1}{2}\rho\boldsymbol{u}_{\mathrm{m}}^2\,\mathrm{d}V \quad \text{and} \quad \mathscr{A}(t) \equiv \int_V \rho A\,\mathrm{d}V,$$

we obtain, on time-integration from an initial state at time $t = 0$,

$$\mathscr{M}(t) + \mathscr{A}(t) = \mathscr{M}(0) + \mathscr{A}(0)$$

and so

$$\mathscr{M}(t) \leqslant \mathscr{M}(0) + \mathscr{A}(0) \tag{6.1}$$

provided that

$$\mathscr{A}(t) \geqslant 0. \tag{6.2}$$

(Note that $\mathscr{A}$ may also be regarded as the *pseudoenergy* of the fluid: see Codoban & Shepherd 2003.)

We have already shown in §4 that if conditions (4.5) hold at $t = 0$ they will hold at later times, for zonally symmetric, adiabatic, frictionless flow. We make the further physically reasonable assumptions that conditions (4.6) and (4.7) also hold for all t. Under these conditions $A(\boldsymbol{x}, t) \geqslant 0$ for all $\boldsymbol{x}$ and t, so that (6.2) is satisfied for all t. Hence the initial state is stable to finite-amplitude disturbances, in the Lyapunov sense that the meridional kinetic energy $\mathscr{M}(t)$ remains bounded for all t, by (6.1). (Note, however, that this argument does not imply that the zonal kinetic energy $\mathscr{K} \equiv \frac{1}{2}\int_V \rho u^2\,\mathrm{d}V$ is bounded: it does not, on its own, exclude the possibility that $\mathscr{K}$ might for example grow monotonically at the expense of total potential energy, without a change in $\mathscr{A}$.)

This situation is similar to, but slightly more complicated than, the zonally symmetric Boussinesq case on a rotating 'f-plane', for which the sole stability condition is $P > 0$, where the potential vorticity P is conserved as the flow evolves. Under adiabatic, frictionless conditions, a flow that is symmetrically stable cannot evolve into one that is symmetrically unstable: see e.g. Hoskins (1974).

The reference state must be a steady solution of the equations of motion, but is otherwise somewhat arbitrary (cf. A81). Since the initial state whose stability is to be investigated must – in any non-trivial case – be one that evolves in time, it cannot itself be chosen as the reference state; it might, however, be a small-amplitude, or even a finite-amplitude, perturbation of a suitably defined reference state.

7. Discussion

The results obtained in this paper may readily be extended to a rotating reference frame and to spherical coordinates; these are particularly relevant for treating models of global-scale axisymmetric motions on a rotating planet. Following the approach introduced by Codoban & Shepherd (2003) for a Boussinesq fluid, the concept of an 'available energy' may also be extended, at least in a formal sense, to treat the zonally averaged flow of a compressible fluid in the presence of non-axisymmetric ('eddy') processes and non-adiabatic and frictional processes: brief details are outlined in the Appendix. This extension is relevant for example to zonally symmetric atmospheric models that are mechanically driven but thermally damped. The causality of the energetics of such models was clarified by Codoban & Shepherd (2003) using a formalism similar to that presented in the Appendix.

It should be noted that Hamiltonian methods have not explicitly been used in this paper, although the results must ultimately stem from the underlying Hamiltonian structure of the conservative case.

It would be interesting to discover whether the available energy concept introduced here could be extended to non-axisymmetric flows, using conservation of mass, specific entropy and potential vorticity (but not specific angular momentum), and replacing $\tilde{C}(\mu, s)$ by a function of P and s. It is not immediately obvious how this might be done, however: such a function does not arise naturally from an argument analogous to that leading to (3.5), since P_0 does not figure directly in the momentum equation for the reference flow.

I acknowledge helpful discussions with T. G. Shepherd on this topic over the past 15 years or so. An anonymous referee kindly drew my attention to the paper by Ilin (1991) and to the requirement that the domain of $C(\mu, s)$ be convex.

Appendix. Inclusion of non-axisymmetric and non-conservative terms

In this appendix we briefly note how the available energy concept is modified in the presence of zonally asymmetric and non-conservative terms, following Codoban & Shepherd (2003). To represent these additional effects we rewrite (2.1)–(2.4) in the form

$$\frac{\mathrm{D_m}\boldsymbol{u}}{\mathrm{D}t} + \frac{1}{\rho}\nabla p + \nabla \Phi = \boldsymbol{X}, \tag{A 1}$$

$$\frac{\mathrm{D_m}\rho}{\mathrm{D}t} + \rho \nabla \cdot \boldsymbol{u} = S, \tag{A 2}$$

and

$$\frac{\mathrm{D_m}s}{\mathrm{D}t} = Q, \tag{A 3}$$

and the equation of state

$$\rho^{-1} = F(s, p) + E, \tag{A 4}$$

where the symbols $\boldsymbol{u}$, ρ, p s and Φ are now taken to represent the *zonal means* of the relevant quantities, and the terms $\boldsymbol{X}$, S, Q and E on the right-hand sides represent the zonal means of the sums of any 'eddy flux' terms and any non-conservative terms such as friction in (A 1), mass sources or sinks in (A 2) and non-adiabatic heating in (A 3), and departures from a 'two-parameter' equation of state in (A 4). The symbol $\mathrm{D_m}/\mathrm{D}t$ represents $\partial/\partial t + \boldsymbol{u}_\mathrm{m} \cdot \nabla$, where $\boldsymbol{u}_\mathrm{m}$ is the zonal mean of the meridional component of the flow.

We assume that an unforced, steady, zonally symmetric reference state $\mathscr{R}_0$ exists, in the form considered in § 3, so the function $\tilde{C}$ can be defined as before (again using analytic continuation if necessary).

It can be shown that the angular momentum equation (2.7) is replaced by

$$\frac{\mathrm{D_m}m}{\mathrm{D}t} = N, \tag{A 5}$$

where N is the sum of a term dependent on the zonal component of $\boldsymbol{X}$ and eddy flux terms. The energy equation (3.11) becomes

$$\rho \frac{\mathrm{D_m}}{\mathrm{D}t}\left\{\tfrac{1}{2}\boldsymbol{u}^2 + H - \frac{p-p_0}{\rho} + \Phi\right\} + \nabla\cdot[(p-p_0)\,\boldsymbol{u}] = Z_1, \tag{A 6}$$

and (4.3) becomes

$$\rho \frac{\mathrm{D_m}}{\mathrm{D}t}\left(\tfrac{1}{2}\boldsymbol{u}_{\mathrm{m}}^2 + A\right) + \nabla\cdot[(p-p_0)\,\boldsymbol{u}] = Z_2, \tag{A 7}$$

where Z_1 and Z_2 depend on $\boldsymbol{X}$, S, Q and E and vanish if they are all zero. The available energy density A is given by the same expression (4.4) as before; it should be emphasized that it is defined in terms of the zonal means of μ, s, p and ρ.

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